

DETERMINATION OF TIME DEPENDENT FACTORS OF COEFFICIENTS IN FRACTIONAL DIFFUSION EQUATIONS

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ABSTRACT. In the present paper, we consider initial-boundary value problems for partial differential equations with time-fractional derivatives which evolve in $Q = \Omega \times (0, T)$ where Ω is a bounded domain of \mathbb{R}^d and $T > 0$. We study the stability of the inverse problem of determining the time-dependent parameter in a source term or a coefficient of zero-th order term from observations of the solution at a point $x_0 \in \overline{\Omega}$ for all $t \in (0, T)$.

1. INTRODUCTION

Let Ω be a bounded domain of \mathbb{R}^d , $d = 1, 2, 3$, with C^2 boundary $\partial\Omega$. We set $\Sigma = \partial\Omega \times (0, T)$ and $Q = \Omega \times (0, T)$. We consider the following two initial-boundary value problem (IBVP in short) for the fractional diffusion equation

$$\begin{cases} \partial_t^\alpha u(x, t) + \mathcal{A}u(x, t) = f(t)R(x, t), & (x, t) \in Q, \\ \mathcal{B}_\sigma u(x, t) = 0, & (x, t) \in \Sigma, \\ u(x, 0) = 0, & x \in \Omega \end{cases} \quad (1.1)$$

and

$$\begin{cases} \partial_t^\alpha v(x, t) + \mathcal{A}v(x, t) + f(t)q(x, t)v(x, t) = 0, & (x, t) \in Q, \\ \mathcal{B}_\sigma v(x, t) = 0, & (x, t) \in \Sigma, \\ v(x, 0) = v_0(x), & x \in \Omega \end{cases} \quad (1.2)$$

with $0 < \alpha < 1$. Here we denote by ∂_t^α the Caputo fractional derivative with respect to t :

$$\partial_t^\alpha g(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{dg}{ds}(s) ds.$$

The differential operator \mathcal{A} is defined by

$$\mathcal{A}u(x, t) := - \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j}(x, t) \right),$$

where the coefficients satisfy

$$a_{ij} = a_{ji}, \quad 1 \leq i, j \leq d, \quad \text{and} \quad \sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j \geq \mu |\xi|^2, \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^d$$

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for some $\mu > 0$. Moreover \mathcal{B}_σ is defined as

$$\mathcal{B}_\sigma u(x) = (1 - \sigma(x))u(x) + \sigma(x)\partial_{\nu_A} u(x), \quad x \in \partial\Omega,$$

where

$$\partial_{\nu_A} u(x) = \sum_{i,j=1}^d a_{ij}(x) \frac{\partial u}{\partial x_i} \nu_j(x)$$

and $\nu = (\nu_1, \dots, \nu_d)$ is the outward normal unit vector to $\partial\Omega$. Here σ is a \mathcal{C}^2 function on $\partial\Omega$ satisfying

$$0 \leq \sigma(x) \leq 1, \quad x \in \partial\Omega.$$

For the regularity of a_{ij} , we assume

$$\begin{cases} a_{ij} \in \mathcal{C}^1(\overline{\Omega}) & \text{if } \sigma \equiv 0, \\ a_{ij} \in \mathcal{C}^2(\overline{\Omega}) & \text{if } \sigma \not\equiv 0. \end{cases}$$

Note that the regularity for a_{ij} depends on whether $\sigma \equiv 0$ or not, which is due to condition (2.3) in the next section.

In the present paper, we consider the inverse problem which consists of determining the function $\{f(t)\}_{t \in (0,T)}$ in (1.1) and (1.2) from the observation of the solution at a point $x_0 \in \overline{\Omega}$ for all $t \in (0, T)$.

The partial differential equations with time fractional derivatives such as (1.1) and (1.2) are proposed as new models describing the anomalous diffusion phenomena. Adams and Gelhar [1] pointed out that the field data in a highly heterogeneous aquifer cannot be described well by the classical advection diffusion equation. Hatano and Hatano [11] applied the continuous-time random walk (CTRW) as a microscopic model of the diffusion of ions in heterogeneous media. From the CTRW model, one can derive a fractional diffusion equation as a macroscopic model (see e.g., Metzler and Klafter [15] and Roman and Alemany [18]). In particular, the fractional diffusion equation can be used as a model for the diffusion of contaminants in a soil. Therefore the inverse problem considered in this paper means the determination of the time evolution of pollution source in (1.1) and reaction rate of pollutants in (1.2) respectively. In this paper, we consider such problems assuming the boundedness of the time-dependent parameter $\{f(t)\}_{t \in (0,T)}$ (see (2.1)).

As monographs of fractional calculus, there are books such as Podlubny [16] and Samko, Kilbas and Marichev [22] for example. As for mathematical works concerned with partial differential equations with time fractional derivatives, we can refer to Agarwal [3], Gejji and Jafari [9], Gorenflo and Mainardi [10], Luchko [14] and references therein.

The remainder of this paper is composed of four sections. In Section 2, we state our main results. In Section 3, we study the forward problem for the IBVPs (1.1) and (1.2) and prove the unique existence and regularity of the solutions. In Sections 4 and 5, we complete the proof of our main results—Theorems 2.1 and 2.2 respectively.

2. MAIN RESULTS

By $L^2(\Omega)$, we denote the usual L^2 -space equipped with the inner product (\cdot, \cdot) and the norm $\|\cdot\| := \|\cdot\|_{L^2(\Omega)}$. Moreover $H^s(\Omega)$, $s \in \mathbb{R}$, and $W^{m,p}(\Omega)$, $m \in \mathbb{N}$, $1 \leq p \leq \infty$, are the Sobolev spaces (see Adams [2] for example).

For the time dependent parameter $\{f(t)\}_{t \in (0,T)}$, we always assume

$$f \in L^\infty(0, T). \quad (2.1)$$

For other given functions in (1.1), we suppose

$$R \in L^p(0, T; H^2(\Omega)), \quad \frac{8}{\alpha} < p \leq \infty \quad \text{and} \quad \mathcal{B}_\sigma R = 0 \quad \text{on } \Sigma. \quad (2.2)$$

On the other hand, in the IBVP (1.2), we suppose

$$\begin{cases} q \in L^\infty(0, T; H^2(\Omega)) & (\text{and } \partial_\nu q = 0 \text{ on } \Sigma \text{ if } \sigma \neq 0), \\ v_0 \in H^4(\Omega) & \text{and } \mathcal{B}_\sigma v_0 = \mathcal{B}_\sigma(\mathcal{A}v_0) = 0 \text{ on } \partial\Omega. \end{cases} \quad (2.3)$$

Assuming these conditions, we prove in Section 3 that the IBVPs (1.1) and (1.2) admit unique solutions $u, v \in \mathcal{C}([0, T]; H^2(\Omega))$ with $\partial_t^\alpha u \in L^p(0, T; H^s(\Omega))$ and $\partial_t^\alpha v \in L^\infty(0, T; H^s(\Omega))$ for some $s > d/2$. Therefore, using the Sobolev embedding theorem (see Theorem 9.8 in Chapter 1 of [13] for example), for any $x_0 \in \overline{\Omega}$, we see that

$$\partial_t^\alpha u(x_0, \cdot) \in L^p(0, T) \quad \text{and} \quad \partial_t^\alpha v(x_0, \cdot) \in L^\infty(0, T).$$

Then our main results can be stated as follows;

Theorem 2.1. *Let condition (2.2) be fulfilled and u_i be the solution of (1.1) for $f = f_i \in L^\infty(0, T)$, ($i = 1, 2$). We assume that there exist $x_0 \in \overline{\Omega}$ and $\delta > 0$ such that*

$$|R(x_0, t)| \geq \delta, \quad \text{a.e. } t \in (0, T). \quad (2.4)$$

Then there exists a constant $C > 0$ depending on p, T, Ω, δ and $\|R\|_{L^p(0,T;H^2(\Omega))}$ such that

$$\|f_1 - f_2\|_{L^p(0,T)} \leq C \|\partial_t^\alpha u_1(x_0, \cdot) - \partial_t^\alpha u_2(x_0, \cdot)\|_{L^p(0,T)}, \quad (2.5)$$

$$\|\partial_t^\alpha u_1(x_0, \cdot) - \partial_t^\alpha u_2(x_0, \cdot)\|_{L^p(0,T)} \leq C \|f_1 - f_2\|_{L^\infty(0,T)}. \quad (2.6)$$

In particular, if we take $p = \infty$ in (2.2), then

$$\begin{aligned} C^{-1} \|\partial_t^\alpha u_1(x_0, \cdot) - \partial_t^\alpha u_2(x_0, \cdot)\|_{L^\infty(0,T)} &\leq \|f_1 - f_2\|_{L^\infty(0,T)} \\ &\leq C \|\partial_t^\alpha u_1(x_0, \cdot) - \partial_t^\alpha u_2(x_0, \cdot)\|_{L^\infty(0,T)}. \end{aligned}$$

Theorem 2.2. *Let condition (2.3) be fulfilled and v_i be the solution of (1.2) for $f = f_i \in L^\infty(0, T)$ with $\|f_i\|_{L^\infty(0,T)} \leq M$ ($i = 1, 2$). We assume that there exist $x_0 \in \overline{\Omega}$ and $\delta > 0$ such that*

$$|q(x_0, t)v_2(x_0, t)| \geq \delta, \quad \text{a.e. } t \in (0, T). \quad (2.7)$$

Then there exists a constant $C > 0$ depending on M, T, Ω, δ and $\|q\|_{L^\infty(0,T;H^2(\Omega))}$ such that

$$C^{-1} \|\partial_t^\alpha v_1(x_0, \cdot) - \partial_t^\alpha v_2(x_0, \cdot)\|_{L^\infty(0,T)} \leq \|f_1 - f_2\|_{L^\infty(0,T)}$$

$$\leq C \|\partial_t^\alpha v_1(x_0, \cdot) - \partial_t^\alpha v_2(x_0, \cdot)\|_{L^\infty(0,T)}. \quad (2.8)$$

In Theorem 4.4 of Sakamoto and Yamamoto [21], a similar problem to Theorem 2.1 is considered, but our result is more applicable in the point of view that the factor $R(x, t)$ is also allowed to depend on t . Moreover, we may assume less regularity for R in Theorem 2.1. The arguments of Theorem 2.2 can also be applied to parabolic equations in order to consider the result of Theorem 1.1 in [7] with observations of the solution at a point $x_0 \in \Omega$ when $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$.

For such inverse problems with $\alpha = 1$, we can also refer to Section 1.5 of Prilepko, Orlovsky and Vasin [17], Cannon and Esteve [6] and Saitoh, Tuan and Yamamoto [19, 20], for example. In our main results, we assume conditions (2.4) and (2.7), which means that the observation point cannot be far from the source. On the other hand, in [6] and [19, 20], the determination of time dependent factor in the source term is considered without assuming such conditions and the logarithmic type and Hölder type estimates are proved respectively. However, the results for fractional diffusion equations without these conditions have not been obtained yet. Here we restrict ourselves to the case with assumptions (2.4) and (2.7), and show the Lipschitz type stability.

Let us remark that the results of this paper can be extended to the case $d \geq 4$. For this purpose additional conditions such as more regularity for a_{ij} and $\partial\Omega$ are required. In order to avoid technical difficulties, we only treat the case $d \leq 3$.

3. FORWARD PROBLEM

This section is devoted to the proof of unique existence and regularity of the solution of the IBVPs (1.1) and (1.2).

Proposition 3.1. *Let conditions (2.1) and (2.2) be fulfilled. Then the IBVP (1.1) admits a unique solution $u \in \mathcal{C}([0, T]; H^2(\Omega))$ satisfying*

$$\mathcal{A}u \in \mathcal{C}([0, T]; H^{2\gamma}(\Omega)) \quad \text{and} \quad \partial_t^\alpha u \in L^p(0, T; H^{2\gamma}(\Omega))$$

for all $0 \leq \gamma < 1 - 1/(p\alpha)$. Moreover we have

$$\|\mathcal{A}u\|_{\mathcal{C}([0, T]; H^{2\gamma}(\Omega))} + \|\partial_t^\alpha u\|_{L^p(0, T; H^{2\gamma}(\Omega))} \leq C \|fR\|_{L^p(0, T; H^2(\Omega))}. \quad (3.1)$$

with $C > 0$ depending on Ω , T and γ

Proposition 3.2. *Let conditions (2.1) and (2.3) be fulfilled. Then the IBVP (1.2) admits a unique solution $v \in \mathcal{C}([0, T]; H^2(\Omega))$ satisfying*

$$\mathcal{A}v \in \mathcal{C}([0, T]; H^{2\gamma}(\Omega)) \quad \text{and} \quad \partial_t^\alpha v \in L^\infty(0, T; H^{2\gamma}(\Omega))$$

for all $0 \leq \gamma < 1$. Moreover, we have

$$\|\mathcal{A}v\|_{\mathcal{C}([0, T]; H^{2\gamma}(\Omega))} + \|\partial_t^\alpha v\|_{L^\infty(0, T; H^{2\gamma}(\Omega))} \leq C \|v_0\|_{H^4(\Omega)} \quad (3.2)$$

with C depending on Ω , T , $\|f\|_{L^\infty(0, T)}$, $\|q\|_{L^\infty(0, T; H^2(\Omega))}$ and γ .

If all coefficients are independent of time variable t , then we can apply eigenfunction expansion and the problems can be reduced to ordinary differential equations of fractional order (e.g. [21]). However, since we consider the determination of the time dependent factor of coefficients, we apply fixed point theorem to show the unique existence of the solutions to (1.1) and (1.2) as in Beckers and Yamamoto [4].

In order to prove these results, we consider the IBVPs with more general data in the next subsections.

3.1. Intermediate results. We introduce the following IBVPs

$$\begin{cases} \partial_t^\alpha u(x, t) + \mathcal{A}u(x, t) = F(x, t), & (x, t) \in Q, \\ \mathcal{B}_\sigma u(x, t) = 0, & (x, t) \in \Sigma, \\ u(x, 0) = 0, & x \in \Omega, \end{cases} \quad (3.3)$$

$$\begin{cases} \partial_t^\alpha v(x, t) + \mathcal{A}v(x, t) + p(x, t)v(x, t) = F(x, t), & (x, t) \in Q, \\ \mathcal{B}_\sigma v(x, t) = 0, & (x, t) \in \Sigma, \\ v(x, 0) = 0, & x \in \Omega, \end{cases} \quad (3.4)$$

and

$$\begin{cases} \partial_t^\alpha v(x, t) + \mathcal{A}v(x, t) + p(x, t)v(x, t) = 0, & (x, t) \in Q, \\ \mathcal{B}_\sigma v(x, t) = 0, & (x, t) \in \Sigma, \\ v(x, 0) = v_0(x), & x \in \Omega. \end{cases} \quad (3.5)$$

We also consider the following conditions

$$F \in L^p(0, T; H^2(\Omega)), \quad \frac{8}{\alpha} < p \leq \infty \quad \text{and} \quad \mathcal{B}_\sigma F = 0 \quad \text{on } \Sigma \quad (3.6)$$

and

$$\begin{cases} 1) \ p \in L^\infty(0, T; H^2(\Omega)) \quad (\text{and } \partial_\nu p = 0 \text{ on } \Sigma \text{ if } \sigma \neq 0), \\ 2) \ v_0 \in H^4(\Omega) \quad \text{and} \quad \mathcal{B}_\sigma v_0 = \mathcal{B}_\sigma(\mathcal{A}v_0) = 0 \quad \text{on } \partial\Omega. \end{cases} \quad (3.7)$$

Note that if we set $F(x, t) = f(t)R(x, t)$, then conditions (2.1) and (2.2) are equivalent to (3.6). Similarly, if we assume $p(x, t) = f(t)q(x, t)$, then conditions (2.1) and (2.3) are equivalent to (3.7). Now let us consider the following intermediate results.

Lemma 3.3. *Let condition (3.6) be fulfilled. Then the IBVP (3.3) admits a unique solution $u \in \mathcal{C}([0, T]; H^2(\Omega))$ satisfying*

$$\mathcal{A}u \in \mathcal{C}([0, T]; H^{2\gamma}(\Omega)) \quad \text{and} \quad \partial_t^\alpha u \in L^p(0, T; H^{2\gamma}(\Omega))$$

for all $0 \leq \gamma < 1 - 1/(p\alpha)$. Moreover we have

$$\|\mathcal{A}u\|_{\mathcal{C}([0, T]; H^{2\gamma}(\Omega))} + \|\partial_t^\alpha u\|_{L^p(0, T; H^{2\gamma}(\Omega))} \leq C\|F\|_{L^p(0, T; H^2(\Omega))} \quad (3.8)$$

with $C > 0$ depending on Ω , T and γ .

Lemma 3.4. *Let $F \in L^\infty(0, T; H^2(\Omega))$ satisfy $\mathcal{B}_\sigma F = 0$ and condition 1) of (3.7) be fulfilled. Then the IBVP (3.4) admits a unique solution $v \in \mathcal{C}([0, T]; H^2(\Omega))$ satisfying*

$$\mathcal{A}v \in \mathcal{C}([0, T]; H^{2\gamma}(\Omega)) \quad \text{and} \quad \partial_t^\alpha v \in L^\infty(0, T; H^{2\gamma}(\Omega))$$

for all $0 \leq \gamma < 1$. Moreover, we have

$$\|\mathcal{A}v\|_{\mathcal{C}([0, T]; H^{2\gamma}(\Omega))} + \|\partial_t^\alpha v\|_{L^\infty(0, T; H^{2\gamma}(\Omega))} \leq C\|F\|_{L^\infty(0, T; H^2(\Omega))} \quad (3.9)$$

with C depending on Ω , T , $\|p\|_{L^\infty(0, T; H^2(\Omega))}$ and γ .

Lemma 3.5. *Let condition (3.7) be fulfilled. Then the IBVP (3.5) admits a unique solution $v \in \mathcal{C}([0, T]; H^2(\Omega))$ satisfying*

$$\mathcal{A}v \in \mathcal{C}([0, T]; H^{2\gamma}(\Omega)) \quad \text{and} \quad \partial_t^\alpha v \in L^\infty(0, T; H^{2\gamma}(\Omega))$$

for all $0 \leq \gamma < 1$. Moreover, we have

$$\|\mathcal{A}v\|_{\mathcal{C}([0, T]; H^{2\gamma}(\Omega))} + \|\partial_t^\alpha v\|_{L^\infty(0, T; H^{2\gamma}(\Omega))} \leq C\|v_0\|_{H^4(\Omega)} \quad (3.10)$$

with C depending on Ω , T , $\|p\|_{L^\infty(0, T; H^2(\Omega))}$ and γ .

From these three lemmata we deduce easily Propositions 3.1 and 3.2.

3.2. Preliminary. We define the operator A as $\mathcal{A} + 1$ in $L^2(\Omega)$ equipped with the boundary condition $\mathcal{B}_\sigma h = 0$;

$$\begin{cases} D(A) := \{h \in H^2(\Omega); \mathcal{B}_\sigma h = 0 \text{ on } \partial\Omega\}, \\ Ah := \mathcal{A}h + h, \quad h \in D(A). \end{cases} \quad (3.11)$$

Then A is a selfadjoint and strictly positive operator with an orthonormal basis of eigenfunctions $(\phi_n)_{n \geq 1}$ of finite order associated to an increasing sequence of eigenvalues $(\lambda_n)_{n \geq 1}$. We can define the fractional power A^γ , $\gamma \geq 0$, of A by

$$\begin{aligned} D(A^\gamma) &:= \left\{ h \in L^2(\Omega); \sum_{n=1}^{\infty} \lambda_n^{2\gamma} |(h, \phi_n)|^2 < \infty \right\}, \\ A^\gamma h &:= \sum_{n=1}^{\infty} \lambda_n^\gamma (h, \phi_n) \phi_n, \quad h \in D(A^\gamma). \end{aligned} \quad (3.12)$$

Then $D(A^\gamma)$ is a Hilbert space with the norm $\|\cdot\|_{D(A^\gamma)}$ defined by $\|h\|_{D(A^\gamma)} := \|A^\gamma h\|$. Since $D(A)$ is continuously embedded into $H^2(\Omega)$ with norm equivalence (see Theorem 5.4 in Chapter 2 of [13] for example), we see by interpolation that

$$\begin{aligned} D(A^\gamma) &\subset H^{2\gamma}(\Omega), \\ C^{-1}\|h\|_{H^{2\gamma}(\Omega)} &\leq \|h\|_{D(A^\gamma)} \leq C\|h\|_{H^{2\gamma}(\Omega)}, \quad h \in D(A^\gamma) \end{aligned}$$

for $0 \leq \gamma \leq 1$.

In order to prepare for the arguments used in this paper, we consider the following Cauchy problem in $L^2(\Omega)$;

$$\begin{cases} \partial_t^\alpha u(t) + Au(t) = F(t), & t \in (0, T), \\ u(0) = 0. \end{cases} \quad (3.13)$$

We define the operator valued function $\{S_A(t)\}_{t \geq 0}$ by

$$S_A(t)h = \sum_{n=1}^{\infty} (h, \phi_n) E_{\alpha,1}(-\lambda_n t^\alpha) \phi_n, \quad h \in L^2(\Omega), \quad t \geq 0,$$

with $E_{\alpha,\beta}$, $\alpha > 0, \beta \in \mathbb{R}$, the Mittag-Leffler function given by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$

Recall that $S_A(t) \in W^{1,1}(0, T; \mathcal{B}(L^2(\Omega)))$ (e.g. [4] and [21]). Moreover, similarly to Theorem 2.2 in [21], for $F \in L^\infty(0, T; L^2(\Omega))$, problem (3.13) admits a unique solution given by

$$u(t) = \int_0^t A^{-1} S'_A(t-s) F(s) ds. \quad (3.14)$$

This solution is lying in $L^\infty(0, T; D(A^\gamma))$ for $0 \leq \gamma < 1$, and, in view of Theorem 1 in [4], we have

$$\|A^{\gamma-1} S'_A(t)h\| \leq C t^{\alpha(1-\gamma)-1} \|h\|, \quad h \in L^2(\Omega), \quad t > 0. \quad (3.15)$$

In particular, the mapping $t \mapsto A^{-1} S'_A(t)$ belongs to $L^q(0, T; \mathcal{B}(L^2(\Omega)))$ if $q \in (1, \infty)$ satisfy $q(1-\alpha) < 1$. Now we apply the following Young's inequality to (3.14);

Lemma 3.6. *Let $f \in L^p(0, T)$ and $g \in L^q(0, T)$ with $1 \leq p, q \leq \infty$ and $1/p + 1/q = 1$. Then the function $f * g$ defined by*

$$f * g(t) := \int_0^t f(t-s)g(s)ds$$

belongs to $\mathcal{C}[0, T]$ and satisfies

$$|f * g(t)| \leq \|f\|_{L^p(0,t)} \|g\|_{L^q(0,t)}, \quad t \in [0, T].$$

Proof. Let \tilde{f} and \tilde{g} be defined by

$$\tilde{f}(t) := \begin{cases} f(t), & t \in (0, T), \\ 0, & t \notin (0, T), \end{cases} \quad \text{and} \quad \tilde{g}(t) := \begin{cases} g(t), & t \in (0, T), \\ 0, & t \notin (0, T). \end{cases}$$

Then applying the Young's inequality for functions on \mathbb{R} (see Exercise 4.30 in Brezis [5] or Appendix A in Stein [23] for example), we obtain the desired result. \square

Let $p \in (1, \infty]$ be as in (3.6). Noting that \mathcal{A} and $A^{-1}S'_A(t)$ commute, we see that for $F \in L^p(0, T; D(A))$,

$$\mathcal{A}u(t) = \int_0^t A^{-1}S'_A(t-s)\mathcal{A}F(s)ds.$$

By $p > 1/\alpha$ and (3.15), the mapping $t \mapsto A^{-1}S'_A(t)$ belongs to $L^q(0, T; \mathcal{B}(L^2(\Omega)))$ where $q \in [1, \infty)$ satisfies $1/p + 1/q = 1$. Therefore by Lemma 3.6, u belongs to $\mathcal{C}([0, T]; D(A))$ and satisfies

$$\|\mathcal{A}u(t)\| \leq \int_0^t \|A^{-1}S'_A(t-s)\| \|\mathcal{A}F(s)\| ds \leq C \int_0^t (t-s)^{\alpha-1} \|F(s)\|_{D(A)} ds \quad (3.16)$$

$$\leq C \left(\int_0^t s^{(\alpha-1)q} ds \right)^{1/q} \|F\|_{L^p(0, t; D(A))} \leq Ct^{\alpha-1/p} \|F\|_{L^p(0, T; D(A))}. \quad (3.17)$$

Thus we can define the map $\mathcal{H} : L^p(0, T; D(A)) \rightarrow \mathcal{C}([0, T]; D(A))$ by

$$\mathcal{H}(w)(t) := \int_0^t A^{-1}S'_A(t-s)w(s)ds, \quad w \in L^p(0, T; D(A)). \quad (3.18)$$

By using these estimates, we will show the unique existence of the solution applying the fixed point theorem.

3.3. Proof of Lemmata 3.3-3.5.

Proof of Lemma 3.3. Let A be the operator defined by (3.11), then the IBVP (3.3) can be rewritten as

$$\begin{cases} \partial_t^\alpha u(t) + Au(t) = u(t) + F(t), & t \in (0, T), \\ u(0) = 0, \end{cases} \quad (3.19)$$

where $u(t) := u(\cdot, t)$ and $F(t) := F(\cdot, t)$. Noting that $F \in L^p(0, T; D(A))$ by (3.6), we see from (3.14) that the solution u of (3.19) satisfies

$$u(t) = \mathcal{H}(u)(t) + \mathcal{H}(F)(t), \quad t \in (0, T),$$

where the map \mathcal{H} is defined by (3.18). Therefore we will look for a fixed point of the map $\mathcal{G} : \mathcal{C}([0, T]; D(A)) \rightarrow \mathcal{C}([0, T]; D(A))$ defined by

$$\mathcal{G}(w)(t) := \mathcal{H}(w)(t) + \mathcal{H}(F)(t), \quad w \in \mathcal{C}([0, T]; D(A)), \quad t \in (0, T). \quad (3.20)$$

By (3.16), for $w \in \mathcal{C}([0, T]; D(A))$, we have

$$\begin{aligned} \|\mathcal{H}(w)(t)\|_{D(A)} &\leq C \int_0^t (t-s)^{\alpha-1} \|w(s)\|_{D(A)} ds \leq C \left(\int_0^t (t-s)^{\alpha-1} ds \right) \|w\|_{\mathcal{C}([0, T]; D(A))} \\ &= \frac{Ct^\alpha}{\alpha} \|w\|_{\mathcal{C}([0, T]; D(A))}. \end{aligned}$$

Repeating the similar calculation, we get

$$\begin{aligned} \|\mathcal{H}^2 w(t)\|_{D(A)} &\leq C \int_0^t (t-s)^{\alpha-1} \|\mathcal{H}w(s)\|_{D(A)} ds \leq \frac{C}{\alpha} \left(\int_0^t (t-s)^{\alpha-1} s^\alpha ds \right) \|w\|_{\mathcal{C}([0, T]; D(A))} \\ &= \frac{C(\Gamma(\alpha)t^\alpha)^2}{\Gamma(2\alpha+1)} \|w\|_{\mathcal{C}([0, T]; D(A))}. \end{aligned}$$

By induction, we have

$$\|\mathcal{H}^n w(t)\|_{D(A)} \leq \frac{C(\Gamma(\alpha)t^\alpha)^n}{\Gamma(n\alpha+1)} \|w\|_{\mathcal{C}([0,T];D(A))}, \quad w \in \mathcal{C}([0,T];D(A)). \quad (3.21)$$

Therefore we obtain

$$\begin{aligned} \|\mathcal{G}^n(w_1) - \mathcal{G}^n(w_2)\|_{\mathcal{C}([0,T];D(A))} &= \|\mathcal{H}^n(w_1 - w_2)\|_{\mathcal{C}([0,T];D(A))} \\ &\leq \frac{C(\Gamma(\alpha)T^\alpha)^n}{\Gamma(n\alpha+1)} \|w_1 - w_2\|_{\mathcal{C}([0,T];D(A))} \end{aligned}$$

for $w_1, w_2 \in \mathcal{C}([0,T];D(A))$. Since \mathcal{G}^n is a contraction for sufficiently large $n \in \mathbb{N}$, \mathcal{G} admits a unique fixed point $u \in \mathcal{C}([0,T];D(A)) \subset \mathcal{C}([0,T];H^2(\Omega))$. Moreover we have

$$u = \mathcal{G}(u) = \mathcal{G}^n(u) = \mathcal{H}^n(u) + \sum_{k=1}^n \mathcal{H}^k(F)$$

for any $n \in \mathbb{N}$. Now we estimate each $\mathcal{H}^k(F)$. First, by (3.17), we have

$$\|\mathcal{H}(F)(t)\|_{D(A)} \leq Ct^{\alpha-1/p} \|F\|_{L^p(0,T;D(A))}.$$

Next we apply (3.16) to have

$$\begin{aligned} \|\mathcal{H}^2(F)(t)\|_{D(A)} &\leq C \int_0^t (t-s)^{\alpha-1} \|\mathcal{H}(F)(s)\|_{D(A)} ds \\ &\leq C \left(\int_0^t (t-s)^{\alpha-1} s^{\alpha-1/p} ds \right) \|F\|_{L^p(0,T;D(A))} \\ &= \frac{C\Gamma(\alpha)\Gamma(\alpha+1-1/p)}{\Gamma(2\alpha+1-1/p)} t^{2\alpha-1/p} \|F\|_{L^p(0,T;D(A))} \\ &\leq \frac{C\Gamma(\alpha)t^{2\alpha-1/p}}{\Gamma(2\alpha+1-1/p)} \|F\|_{L^p(0,T;D(A))}. \end{aligned}$$

Repeating the similar calculation,

$$\begin{aligned} \|\mathcal{H}^3(F)(t)\|_{D(A)} &\leq C \int_0^t (t-s)^{\alpha-1} \|\mathcal{H}^2(F)(s)\|_{D(A)} ds \\ &\leq \frac{C\Gamma(\alpha)}{\Gamma(2\alpha+1-1/p)} \left(\int_0^t (t-s)^{\alpha-1} s^{2\alpha-1/p} ds \right) \|F\|_{L^p(0,T;D(A))} \\ &= \frac{C\Gamma(\alpha)^2 t^{3\alpha-1/p}}{\Gamma(3\alpha+1-1/p)} \|F\|_{L^p(0,T;D(A))}. \end{aligned}$$

By induction, we obtain

$$\|\mathcal{H}^k(F)\|_{\mathcal{C}([0,T];D(A))} \leq \frac{C\Gamma(\alpha)^{k-1} T^{k\alpha-1/p}}{\Gamma(k\alpha+1-1/p)} \|F\|_{L^p(0,T;D(A))}.$$

Therefore

$$\|u\|_{\mathcal{C}([0,T];D(A))} \leq \|\mathcal{H}^n(u)\|_{\mathcal{C}([0,T];D(A))} + \sum_{k=1}^n \|\mathcal{H}^k(F)\|_{\mathcal{C}([0,T];D(A))}$$

$$\leq \frac{C(\Gamma(\alpha)T^\alpha)^n}{\Gamma(n\alpha+1)} \|u\|_{\mathcal{C}([0,T];D(A))} + \sum_{k=1}^n \frac{C\Gamma(\alpha)^{k-1}T^{k\alpha-1/p}}{\Gamma(k\alpha+1-1/p)} \|F\|_{L^p(0,T;D(A))}$$

and by taking sufficiently large $n \in \mathbb{N}$, we obtain

$$\|u\|_{\mathcal{C}([0,T];D(A))} \leq C\|F\|_{L^p(0,T;D(A))} \quad (3.22)$$

with C depending on T and Ω .

Now fix $0 \leq \gamma < 1 - 1/(p\alpha)$. Then for all $t \in (0, T)$, we have $\mathcal{A}u(t) \in D(A^\gamma)$ with

$$A^\gamma(\mathcal{A}u)(t) = \int_0^t A^{\gamma-1}S'_A(t-s)(\mathcal{A}u(s) + \mathcal{A}F(s))ds$$

and by (3.15), we have

$$\|A^{\gamma-1}S'_A(t)\|_{\mathcal{B}(L^2(\Omega))} \leq Ct^{\mu-1},$$

where $\mu := \alpha(1-\gamma)$. Since $\mu > 1/p$, the mapping $t \mapsto A^{\gamma-1}S'_A(t)$ belongs to $L^q(0, T; \mathcal{B}(L^2(\Omega)))$ where $q \in [1, \infty)$ satisfies $1/p + 1/q = 1$. Therefore $\mathcal{A}u$ belongs to $\mathcal{C}([0, T]; D(A^\gamma))$ and

$$\begin{aligned} \|\mathcal{A}u(t)\|_{D(A^\gamma)} &= \left\| \int_0^t A^{\gamma-1}S'_A(t-s)(\mathcal{A}u(s) + \mathcal{A}F(s))ds \right\| \\ &\leq C \int_0^t (t-s)^{\mu-1} \|u(s)\|_{D(A)} ds + C \int_0^t (t-s)^{\mu-1} \|F(s)\|_{D(A)} ds \\ &\leq C \left(\int_0^t (t-s)^{\mu-1} ds \right) \|u\|_{\mathcal{C}([0,T];D(A))} + C \left(\int_0^t s^{q(\mu-1)} ds \right)^{1/q} \|F\|_{L^p(0,t;D(A))} \\ &\leq CT^\mu \|u\|_{\mathcal{C}([0,T];D(A))} + CT^{\mu-1/p} \|F\|_{L^p(0,T;D(A))} \end{aligned} \quad (3.23)$$

Combining this with (3.22), we have

$$\|\mathcal{A}u(t)\|_{D(A^\gamma)} \leq C\|F\|_{L^p(0,T;D(A))} \leq C\|F\|_{L^p(0,T;H^2(\Omega))}.$$

Hence we deduce that $\mathcal{A}u \in \mathcal{C}([0, T]; H^{2\gamma}(\Omega))$ and

$$\|\mathcal{A}u\|_{\mathcal{C}([0,T];H^{2\gamma}(\Omega))} \leq C\|F\|_{L^p(0,T;H^2(\Omega))}.$$

By the original equation $\partial_t^\alpha u = -\mathcal{A}u + F$, we see that $\partial_t^\alpha u$ belongs to $L^p(0, T; H^{2\gamma}(\Omega))$ with the estimate;

$$\begin{aligned} \|\partial_t^\alpha u\|_{L^p(0,T;H^{2\gamma}(\Omega))} &\leq C\|\mathcal{A}u\|_{L^p(0,T;H^{2\gamma}(\Omega))} + C\|F\|_{L^p(0,T;H^{2\gamma}(\Omega))} \\ &\leq C\|\mathcal{A}u\|_{\mathcal{C}([0,T];H^{2\gamma}(\Omega))} + C\|F\|_{L^p(0,T;H^2(\Omega))} \\ &\leq C\|F\|_{L^p(0,T;H^2(\Omega))}, \end{aligned}$$

which implies (3.8). Thus we have completed the proof. \square

For the proof of Lemma 3.4, we prepare the following fact;

Lemma 3.7. *Let $u, v \in H^2(\Omega)$ and $d \leq 3$, then $uv \in H^2(\Omega)$ with the estimate*

$$\|uv\|_{H^2(\Omega)} \leq C\|v\|_{H^2(\Omega)}$$

with C depending on $\|u\|_{H^2(\Omega)}$.

For this lemma, see Theorem 2.1 in Chapter II of Strichartz [24].

Proof of Lemma 3.4. Similarly to Lemma 3.3, the IBVP (3.4) can be rewritten as

$$\begin{cases} \partial_t^\alpha v(t) + Av(t) = (1 - p(t))v(t) + F(t), \\ v(0) = 0, \end{cases} \quad (3.24)$$

where $v(t) := v(\cdot, t)$ and $F(t) := F(\cdot, t)$. Moreover $p(t)$ denotes the multiplication operator by $p(x, t)$. Then we can see that the solution v of (3.24) is a fixed point of the map $\mathcal{K} : \mathcal{C}([0, T]; D(A)) \rightarrow \mathcal{C}([0, T]; D(A))$ defined by

$$\mathcal{K}(w)(t) := (\mathcal{H}(1 - p(t))w)(t) + \mathcal{H}(F)(t), \quad w \in \mathcal{C}([0, T]; D(A)), \quad t \in (0, T).$$

Indeed, Lemma 3.7 and condition 1) of (3.7) yields that $(1 - p)w$ belongs to $L^\infty(0, T; D(A))$ and satisfies

$$\|(1 - p(t))w(t)\|_{D(A)} \leq C\|w(t)\|_{D(A)}$$

with C depending on $\|p\|_{L^\infty(0, T; H^2(\Omega))}$. Therefore we can see that \mathcal{K} maps $\mathcal{C}([0, T]; D(A))$. Moreover, by the similar calculation to (3.21), we have

$$\|(\mathcal{H}(1 - p))^n(w)\|_{\mathcal{C}([0, T]; D(A))} \leq \frac{C(\Gamma(\alpha)t^\alpha)^n}{\Gamma(n\alpha + 1)}\|w\|_{\mathcal{C}([0, T]; D(A))}, \quad w \in \mathcal{C}([0, T]; D(A)) \quad (3.25)$$

and

$$\|(\mathcal{H}(1 - p))^{n-1}(\mathcal{H}F)\|_{\mathcal{C}([0, T]; D(A))} \leq \frac{C(\Gamma(\alpha)t^\alpha)^n}{\Gamma(n\alpha + 1)}\|F\|_{L^\infty(0, T; D(A))}, \quad F \in L^\infty(0, T; D(A)). \quad (3.26)$$

By (3.25), we find

$$\begin{aligned} \|\mathcal{K}^n(w_1) - \mathcal{K}^n(w_2)\|_{\mathcal{C}([0, T]; D(A))} &\leq \frac{C(\Gamma(\alpha)T^\alpha)^n}{\Gamma(n\alpha + 1)}\|w_1 - w_2\|_{\mathcal{C}([0, T]; D(A))}, \\ w_1, w_2 &\in \mathcal{C}([0, T]; D(A)), \end{aligned}$$

which implies that \mathcal{K} admits a unique fixed point $v \in \mathcal{C}([0, T]; D(A)) \subset \mathcal{C}([0, T]; H^2(\Omega))$. Then we have

$$v = \mathcal{K}(v) = \mathcal{K}^n(v) = (\mathcal{H}(1 - p(t)))^n(v) + \sum_{k=1}^n (\mathcal{H}(1 - p(t)))^{k-1}(\mathcal{H}F). \quad (3.27)$$

Repeating the argument in the proof of Lemma 3.3, we deduce from (3.25), (3.26) and (3.27) that

$$\|v\|_{\mathcal{C}([0, T]; D(A))} \leq C\|F\|_{L^\infty(0, T; D(A))} \quad (3.28)$$

with C depending on T , Ω and $\|p\|_{L^\infty(0, T; H^2(\Omega))}$.

Next we fix $0 \leq \gamma < 1$. Similarly to (3.23), we have

$$\mathcal{A}v(t) \in D(A^\gamma), \quad t \in (0, T)$$

and

$$\|\mathcal{A}v(t)\|_{D(A^\gamma)} \leq C \left\| \int_0^t A^{\gamma-1} S'_A(t-s) ((\mathcal{A}(1 - p(s))v)(s) + \mathcal{A}F(s)) ds \right\|$$

$$\begin{aligned}
&\leq C \int_0^t (t-s)^{\mu-1} (\|(1-p(s))v(s)\|_{D(A)} + \|F(s)\|_{D(A)}) ds \\
&\leq C \int_0^t (t-s)^{\mu-1} (\|v(s)\|_{D(A)} + \|F(s)\|_{D(A)}) ds
\end{aligned}$$

with $\mu = \alpha(1-\gamma)$. Therefore $\mathcal{A}v$ belongs to $\mathcal{C}([0, T]; H^{2\gamma}(\Omega))$ and satisfies

$$\begin{aligned}
\|\mathcal{A}v\|_{\mathcal{C}([0, T]; H^{2\gamma}(\Omega))} &\leq \|\mathcal{A}v\|_{\mathcal{C}([0, T]; D(A^\gamma))} \leq CT^\mu (\|v\|_{\mathcal{C}([0, T]; D(A))} + \|F\|_{L^\infty(0, T; D(A))}) \\
&\leq C\|F\|_{L^\infty(0, T; D(A))} \leq C\|F\|_{L^\infty(0, T; H^2(\Omega))},
\end{aligned}$$

where we have used (3.28). Moreover, combining this with the original equation, we also have $\partial_t^\alpha v \in L^\infty(0, T; H^{2\gamma}(\Omega))$ and (3.9). \square

Proof of Lemma 3.5. We split the solution v of (3.5) into two terms $v = w + v_0$ where w solves

$$\begin{cases} \partial_t^\alpha w(x, t) + \mathcal{A}w(x, t) + p(x, t)w(x, t) = F(x, t), & (x, t) \in Q, \\ \mathcal{B}_\sigma w(x, t) = 0, & (x, t) \in \Sigma, \\ w(x, 0) = 0, & x \in \Omega \end{cases} \quad (3.29)$$

with $F(x, t) := -(\mathcal{A} + p(x, t))v_0(x)$. Then (3.7) implies $F \in L^\infty(0, T; D(A))$ with the estimate

$$\|F\|_{L^\infty(0, T; H^2(\Omega))} \leq C\|v_0\|_{H^4(\Omega)}.$$

By Lemma 3.4, the IBVP (3.29) admits a unique solution $w \in \mathcal{C}([0, T]; H^2(\Omega))$ satisfying

$$\mathcal{A}w \in \mathcal{C}([0, T]; H^{2\gamma}(\Omega)) \quad \text{and} \quad \partial_t^\alpha w \in L^\infty(0, T; H^{2\gamma}(\Omega)).$$

Moreover

$$\|\mathcal{A}w\|_{\mathcal{C}([0, T]; H^{2\gamma}(\Omega))} + \|\partial_t^\alpha w\|_{L^\infty(0, T; H^{2\gamma}(\Omega))} \leq C\|F\|_{L^\infty(0, T; H^2(\Omega))} \leq C\|v_0\|_{H^4(\Omega)}.$$

Therefore the IBVP (3.5) admits a unique solution $v \in \mathcal{C}([0, T]; H^2(\Omega))$ satisfying

$$\mathcal{A}v \in \mathcal{C}([0, T]; H^{2\gamma}(\Omega)) \quad \text{and} \quad \partial_t^\alpha v \in L^\infty(0, T; H^{2\gamma}(\Omega)).$$

From the above estimate, we deduce (3.10). \square

4. PROOF OF THEOREM 2.1

In this section, we prove Theorem 2.1. To this end, we prepare the following lemmata with Gronwall type inequalities;

Lemma 4.1. *Let $C, \alpha > 0$ and $u, d \in L^1(0, T)$ be nonnegative functions satisfying*

$$u(t) \leq Cd(t) + C \int_0^t (t-s)^{\alpha-1} u(s) ds, \quad t \in (0, T),$$

then we have

$$u(t) \leq Cd(t) + C \int_0^t (t-s)^{\alpha-1} d(s) ds, \quad t \in (0, T).$$

For the proof, see Lemma 7.1.1 p.188 of [12].

Lemma 4.2. *We take $2 \leq p \leq \infty$ and $\mu > 2/p$. Let $f \in L^\infty(0, T)$ and $u, R \in L^p(0, T)$ be non-negative functions satisfying the integral inequality*

$$f(t) \leq u(t) + \int_0^t (t-s)^{\mu-1} f(s) R(s) ds, \quad a.e. t \in (0, T). \quad (4.1)$$

Then we have

$$\|f\|_{L^p(0, T)} \leq C \|u\|_{L^p(0, T)}, \quad (4.2)$$

where the constant C depends on p, μ, T and $\|R\|_{L^p(0, T)}$.

Proof. We set $d(t) := \|f\|_{L^p(0, t)}^p$. From equation (4.1), we have

$$|f(s)|^p \leq C |u(s)|^p + C \left| \int_0^s (s-\xi)^{\mu-1} f(\xi) R(\xi) d\xi \right|^p,$$

which implies

$$d(t) \leq C \|u\|_{L^p(0, T)}^p + C \int_0^t \left| \int_0^s (s-\xi)^{\mu-1} f(\xi) R(\xi) d\xi \right|^p ds. \quad (4.3)$$

Now we estimate the right-hand side of the above. By the Cauchy-Schwarz inequality,

$$\int_0^s |f(\xi) R(\xi)|^{p/2} d\xi = \int_0^s |f(\xi)|^{p/2} \cdot |R(\xi)|^{p/2} d\xi \leq \left(\int_0^s |f(\xi)|^p d\xi \right)^{1/2} \left(\int_0^s |R(\xi)|^p d\xi \right)^{1/2},$$

that is,

$$\|fR\|_{L^{p/2}(0, s)} \leq \|f\|_{L^p(0, s)} \|R\|_{L^p(0, s)}.$$

Therefore if $p > 2$, then Lemma 3.6 yields that

$$\left| \int_0^s (s-\xi)^{\mu-1} f(\xi) R(\xi) d\xi \right| \leq \left(\int_0^s \xi^{r(\mu-1)} ds \right)^{1/r} \|fR\|_{L^{p/2}(0, s)} \leq C \|f\|_{L^p(0, s)} \|R\|_{L^p(0, s)},$$

where $r \in [1, \infty)$ satisfies $2/p + 1/r = 1$. For $p = 2$, we also have

$$\left| \int_0^s (s-\xi)^{\mu-1} f(\xi) R(\xi) d\xi \right| \leq s^{\mu-1} \|fR\|_{L^1(0, s)} \leq C \|f\|_{L^2(0, s)} \|R\|_{L^2(0, s)}.$$

Thus for any $p \geq 2$, we have

$$\left| \int_0^s (s-\xi)^{\mu-1} f(\xi) R(\xi) d\xi \right| \leq C d(s), \quad (4.4)$$

where C depends on T, p, μ and $\|R\|_{L^p(0, T)}$. By (4.3) and (4.4), we have

$$d(t) \leq C \|u\|_{L^p(0, T)}^p + C \int_0^t d(s) ds, \quad t \in (0, T).$$

Hence by the Gronwall inequality, we have

$$d(t) \leq C \|u\|_{L^p(0, T)}^p, \quad t \in (0, T)$$

with C depending on p, μ, T and $\|R\|_{L^p(0, T)}$. Thus we have proved (4.2). \square

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Let u_i be the solutions to (1.1) corresponding to f_i ($i = 1, 2$) and set $u := u_1 - u_2$ and $f := f_1 - f_2$. Then u solves (1.1) and is given by

$$u(t) = \int_0^t A^{-1} S'_A(t-s) u(s) + \int_0^t A^{-1} S'_A(t-s) f(s) R(s) ds,$$

where $u(t) := u(\cdot, t)$ and $R(t) := R(\cdot, t)$.

First we estimate $\|u(t)\|_{D(A)}$. Similarly to the calculation in (3.16), we have

$$\begin{aligned} \|u(t)\|_{D(A)} &\leq C \int_0^t (t-s)^{\alpha-1} \|u(s)\|_{D(A)} ds + C \int_0^t (t-s)^{\alpha-1} |f(s)| \|R(s)\|_{D(A)} ds \\ &= C \int_0^t (t-s)^{\alpha-1} \|u(s)\|_{D(A)} ds + Cd(t), \end{aligned} \quad (4.5)$$

where we have set

$$d(t) := \int_0^t (t-s)^{\alpha-1} |f(s)| \|R(s)\|_{D(A)} ds.$$

Applying Lemma 4.1 to (4.5), we have

$$\|u(t)\|_{D(A)} \leq Cd(t) + C \int_0^t (t-s)^{\alpha-1} d(s) ds, \quad 0 < t < T. \quad (4.6)$$

Here for $\nu > 0$, we note

$$\begin{aligned} \int_0^t (t-s)^{\nu-1} d(s) ds &= \int_0^t (t-s)^{\nu-1} \left(\int_0^s (s-\xi)^{\alpha-1} |f(\xi)| \|R(\xi)\|_{D(A)} d\xi \right) ds \\ &= \int_0^t \left(\int_\xi^t (t-s)^{\nu-1} (s-\xi)^{\alpha-1} ds \right) |f(\xi)| \|R(\xi)\|_{D(A)} d\xi \\ &= B(\nu, \alpha) \int_0^t (t-\xi)^{\nu+\alpha-1} |f(\xi)| \|R(\xi)\|_{D(A)} d\xi \end{aligned} \quad (4.7)$$

where $B(\cdot, \cdot)$ is the Beta function. In particular, for $\nu = \alpha$, we have

$$\begin{aligned} \int_0^t (t-s)^{\alpha-1} d(s) ds &= B(\alpha, \alpha) \int_0^t (t-s)^{2\alpha-1} |f(s)| \|R(s)\|_{D(A)} ds \\ &\leq T^\alpha B(\alpha, \alpha) \int_0^t (t-s)^{\alpha-1} |f(s)| \|R(s)\|_{D(A)} ds \\ &\leq Cd(t). \end{aligned}$$

Hence the following estimate follows from (4.6);

$$\|u(t)\|_{D(A)} \leq Cd(t), \quad 0 < t < T. \quad (4.8)$$

Next we estimate $\|\mathcal{A}u(t)\|_{D(A^\gamma)}$ for $d/4 < \gamma < 1 - 2/(p\alpha)$. Repeating the calculation in (3.23), we find

$$\|\mathcal{A}u(t)\|_{D(A^\gamma)} \leq C \int_0^t (t-s)^{\mu-1} (\|u(s)\|_{D(A)} + |f(s)| \|R(s)\|_{D(A)}) ds, \quad \text{a.e. } t \in (0, T)$$

where $\mu = \alpha(1 - \gamma)$. By (4.7) with $\nu = \mu$ and (4.8), we obtain

$$\begin{aligned}
\|\mathcal{A}u(t)\|_{D(A^\gamma)} &\leq C \int_0^t (t-s)^{\mu-1} d(s) ds + C \int_0^t (t-s)^{\mu-1} |f(s)| \|R(s)\|_{D(A)} ds \\
&= CB(\mu, \alpha) \int_0^t (t-s)^{\mu+\alpha-1} |f(s)| \|R(s)\|_{D(A)} ds \\
&\quad + C \int_0^t (t-s)^{\mu-1} |f(s)| \|R(s)\|_{D(A)} ds \\
&\leq CT^\alpha B(\mu, \alpha) \int_0^t (t-s)^{\mu-1} |f(s)| \|R(s)\|_{D(A)} ds \\
&\quad + C \int_0^t (t-s)^{\mu-1} |f(s)| \|R(s)\|_{D(A)} ds \\
&\leq C \int_0^t (t-s)^{\mu-1} |f(s)| \|R(s)\|_{D(A)} ds.
\end{aligned}$$

Finally we estimate $|\mathcal{A}u(x_0, t)|$ and complete the proof. Since $\gamma > d/4$, the Sobolev embedding theorem yields

$$|\mathcal{A}u(x_0, t)| \leq C \|\mathcal{A}u(\cdot, t)\|_{H^{2\gamma}(\Omega)} \leq C \|\mathcal{A}u(t)\|_{D(A^\gamma)} \leq C \int_0^t (t-s)^{\mu-1} |f(s)| \|R(s)\|_{D(A)} ds. \quad (4.9)$$

From the original equation, we get

$$f(t)R(x_0, t) = \partial_t^\alpha u(x_0, t) + \mathcal{A}u(x_0, t), \quad \text{a.e. } t \in (0, T). \quad (4.10)$$

Combining this with (2.4) and (4.9), we get

$$\begin{aligned}
|f(t)| &\leq \frac{1}{\delta} (|\partial_t^\alpha u(x_0, t)| + |\mathcal{A}u(x_0, t)|) \\
&\leq C |\partial_t^\alpha u(x_0, t)| + C \int_0^t (t-s)^{\mu-1} |f(s)| \|R(s)\|_{D(A)} ds, \quad \text{a.e. } t \in (0, T)
\end{aligned} \quad (4.11)$$

with C depending on δ , Ω and T . By Lemma 4.2, we see that

$$\|f\|_{L^p(0, T)} \leq C \|\partial_t^\alpha u(x_0, \cdot)\|_{L^p(0, T)},$$

which implies (2.5). Moreover, by (4.9) and (4.10), we have

$$\begin{aligned}
|\partial_t^\alpha u(x_0, t)| &\leq |f(t)R(x_0, t)| + |\mathcal{A}u(x_0, t)| \\
&\leq C |f(t)| \|R(\cdot, t)\|_{H^2(\Omega)} + C \int_0^t (t-s)^{\mu-1} |f(s)| \|R(s)\|_{D(A)} ds \\
&\leq C \|f\|_{L^\infty(0, T)} \|R(t)\|_{D(A)} + C \|f\|_{L^\infty(0, T)} \int_0^t (t-s)^{\mu-1} \|R(s)\|_{D(A)} ds \\
&\leq C \|f\|_{L^\infty(0, T)} \|R(t)\|_{D(A)} + C \|f\|_{L^\infty(0, T)} T^{\mu-1/p} \|R\|_{L^p(0, T; D(A))}, \quad \text{a.e. } t \in (0, T).
\end{aligned}$$

Therefore,

$$\|\partial_t^\alpha u(x_0, \cdot)\|_{L^p(0, T)} \leq C \|f\|_{L^\infty(0, T)} \|R\|_{L^p(0, T; D(A))} + C \|f\|_{L^\infty(0, T)} T^\mu \|R\|_{L^p(0, T; D(A))}$$

$$\leq C\|f\|_{L^\infty(0,T)}.$$

Thus we have proved (2.6). \square

5. PROOF OF THEOREM 2.2

In this section, we prove Theorem 2.2. We first prepare the following generalized Gronwall's inequality;

Lemma 5.1. *Let $\mu, a, b > 0$ and $f \in L^1(0, T)$ be nonnegative function satisfying the integral inequality*

$$f(t) \leq a + b \int_0^t (t-s)^{\mu-1} f(s) ds, \quad a.e. \ t \in (0, T).$$

Then we have

$$f(t) \leq a E_{\mu,1}((b\Gamma(\mu))^{1/\mu} t^\mu), \quad a.e. \ t \in (0, T).$$

For the proof, see Lemma 7.1.2 on p.189 of [12]. Now we are ready to prove Theorem 2.2.

Proof of Theorem 2.2. Let v_i be the solutions to (1.2) corresponding to f_i ($i = 1, 2$) and set $v := v_1 - v_2$ and $f := f_2 - f_1$. Then v solves (3.4) with $p(x, t) = f_1(t)q(x, t)$ and $F(x, t) = f(t)q(x, t)v_2(x, t)$. Recall that v is given by

$$v(t) = \int_0^t A^{-1} S'_A(t-s)((1-p(t))v)(s) + \int_0^t f(s) A^{-1} S'_A(t-s) R(s) ds,$$

where we have set $v(t) := v(\cdot, t)$ and $R(t) := q(\cdot, t)v_2(\cdot, t)$. Moreover, $p(t)$ denotes the multiplication operator by $p(x, t) := f_1(t)q(x, t)$.

First we estimate $\|v(t)\|_{D(A)}$. Since $(1-p(t))v(t), R(t) \in D(A)$ by (2.3), we repeat the calculation in (3.23) to have

$$\begin{aligned} \|v(t)\|_{D(A)} &\leq C \int_0^t (t-s)^{\alpha-1} \|(1-p(t))v(s)\|_{D(A)} ds + C \int_0^t (t-s)^{\alpha-1} |f(s)| \|R(s)\|_{D(A)} ds \\ &\leq C \int_0^t (t-s)^{\alpha-1} \|v(s)\|_{D(A)} ds + C \int_0^t (t-s)^{\alpha-1} |f(s)| ds. \end{aligned}$$

with C depending on Ω , M and $\|q\|_{L^\infty(0,T;H^2(\Omega))}$. Then repeating the arguments used in Theorem 2.1, we obtain

$$\|v(t)\|_{D(A)} \leq C \int_0^t (t-s)^{\alpha-1} |f(s)| ds, \quad 0 < t < T.$$

and from this estimate we also deduce that for any $0 \leq \gamma < 1$,

$$\|\mathcal{A}v(t)\|_{D(A^\gamma)} \leq C \int_0^t (t-s)^{\mu-1} |f(s)| ds, \quad 0 < t < T,$$

where $\mu := \alpha(1-\gamma)$. Therefore by taking $\gamma \in (d/4, 1)$, we have

$$\begin{aligned} |\mathcal{A}v(x_0, t) + p(x_0, t)v(x_0, t)| &\leq C \|\mathcal{A}v(\cdot, t) + p(\cdot, t)v(\cdot, t)\|_{H^{2\gamma}(\Omega)} \\ &\leq C \|\mathcal{A}v(\cdot, t)\|_{H^{2\gamma}(\Omega)} + C \|v(\cdot, t)\|_{H^{2\gamma}(\Omega)} \end{aligned}$$

$$\leq C\|\mathcal{A}v(t)\|_{D(A^\gamma)} \leq C \int_0^t (t-s)^{\mu-1} |f(s)| ds. \quad (5.1)$$

From the original equation, we have

$$f(t)R(x_0, t) = \partial_t^\alpha v(x_0, t) + \mathcal{A}v(x_0, t) + p(x_0, t)v(x_0, t), \quad \text{a.e. } t \in (0, T). \quad (5.2)$$

On the other hand, from (2.7), we deduce that

$$|R(x_0, t)| \geq c > 0, \quad \text{a.e. } t \in (0, T)$$

with c depending on δ , Ω and T . Therefore, combining this with (5.1) and (5.2), we obtain

$$\begin{aligned} |f(t)| &\leq C|\partial_t^\alpha v(x_0, t)| + C|\mathcal{A}v(x_0, t) + p(x_0, t)v(x_0, t)| \\ &\leq C\|\partial_t^\alpha v(x_0, \cdot)\|_{L^\infty(0, T)} + C \int_0^t (t-s)^{\mu-1} |f(s)| ds, \quad \text{a.e. } t \in (0, T). \end{aligned}$$

Applying Lemma 5.1, we see that

$$|f(t)| \leq C\|\partial_t^\alpha v(x_0, \cdot)\|_{L^\infty(0, T)}.$$

Thus we have proved the second inequality in (2.8). Moreover, by (5.2), we have

$$\begin{aligned} |\partial_t^\alpha v(x_0, t)| &\leq |f(t)R(x_0, t)| + |\mathcal{A}v(x_0, t) + p(x_0, t)v(x_0, t)| \\ &\leq |f(t)|\|R(\cdot, t)\|_{D(A)} + C \int_0^t (t-s)^{\mu-1} |f(s)| ds \\ &\leq C \left(\|R\|_{L^\infty(0, T; D(A))} + \frac{T^\mu}{\mu} \right) \|f\|_{L^\infty(0, T)}. \end{aligned}$$

Thus we have proved the first inequality in (2.8). \square

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